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Asymptotic Cones and Duality of Linear Relations*

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1. INTRODUCTION

Let X be a non-empty closed convex set in a separated topological vector space. It is easily seen that for every $x \in X$, the intersection

$$C_X = \bigcap_{\lambda > 0} \lambda(X - x)$$

is a closed convex cone independent of the choice of x in X. Following Choquet [*2*], [*3*], C_X is called the *asymptotic cone* of X. For every $x \in X$, $x + C_X$ is the union of all closed halflines beginning at x and contained in X.

The following result has been recently obtained by Dieudonné [*4*]:

Let X, Y be two non-empty closed convex sets in a separated topological vector space E. If $C_X \cap C_Y = \{0\}$ and if at least one of X, Y is locally compact, then $X - Y$ is closed in E.

In the present paper, we shall study duality of linear relations by using the notion of asymptotic cones. Section 2 is concerned with existence theorems for linear relations involving closed, locally compact convex sets. Section 3 deals with dual extremal problems. All results in this paper are based on Dieudonné's theorem stated above. They improve and sharpen some of the results obtained previously in ([*5*], Theorems 4–7) by a different approach.

All vector spaces considered here are implicitly assumed to be real vector spaces. By a convex cone in a vector space E, we shall understand a convex cone with its vertex at the origin 0 of E. For a topological vector space E, the dual space of E (i.e., the vector space of all continuous linear forms on E) is denoted by E'. For a set $X \subset E$, X^0 and X^{00} denote the polar and bipolar of X [*1*], [*6*]. Thus,

$$X^0 = \{f \in E' : f(x) \leqslant 1 \text{ for all } x \in X\},$$

$$X^{00} = \{x \in E : f(x) \leqslant 1 \text{ for all } f \in X^0\}.$$

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The transpose of a continuous linear map A is denoted by ${}^tA$. The empty set is denoted by $\varnothing$.

2. Existence Theorems

We first state the following result:

THEOREM 1. *Let E, F be two separated, locally convex topological vector spaces. Let $A: E \to F$ be a continuous linear map such that the image $A(E)$ is closed in F. Let K be a closed, locally compact, convex set in F such that $A(E) \cap K \neq \varnothing$ and $A(E) \cap C_K = \{0\}$. Then for any $y_0 \in F$, there exists an $x \in E$ satisfying*

$$y_0 - Ax \in K, \tag{1}$$

if and only if

$$g \in K^0 \text{ and } {}^tAg = 0 \text{ imply } g(y_0) \leqslant 1. \tag{2}$$

Remark 1. The hypothesis $A(E) \cap C_K = \{0\}$ in the above theorem is essential. This can be seen from the following simple example. In the Euclidean 3-space R^3, let K be the closed convex cone formed by all $(\xi_1, \xi_2, \xi_3) \in R^3$ such that $\xi_1 \geqslant 0, \xi_2 \geqslant 0, \xi_1\xi_2 \geqslant \xi_3^2$. Let $A: R^3 \to R^3$ be defined by $A(\xi_1, \xi_2, \xi_3) = (0, \xi_2, 0)$, and let $y_0 = (0, 0, 1)$. We regard R^3 as its own dual space by identifying $g = (\eta_1, \eta_2, \eta_3) \in R^3$ with the linear form $g(x) = \eta_1\xi_1 + \eta_2\xi_2 + \eta_3\xi_3$ for $x = (\xi_1, \xi_2, \xi_3)$. Then K^0 is the set of all $(\eta_1, \eta_2, \eta_3) \in R^3$ such that $\eta_1 \leqslant 0$, $\eta_2 \leqslant 0$ and $\eta_3^2 \leqslant 4\eta_1\eta_2$. Condition (2) is satisfied, but no $x \in E$ satisfies (1). Here Theorem 1 fails to apply, because $A(R^3) \cap C_K \neq \{0\}$.

Theorem 1 is a special case (i.e., the case $f_0 = 0$, $\alpha = 0$) of the following more general result.

THEOREM 2. *Let E, F be two separated, locally convex topological vector spaces. Let $A: E \to F$ be a continuous linear map such that $A(E)$ is closed in F. Let K be a closed, locally compact, convex set in F and $f_0 \in {}^tA(F')$. Suppose that the following conditions* (3), (4) *are fulfilled:*

$$\text{There is an } x_0 \in E \text{ such that } Ax_0 \in K \text{ and } f_0(x_0) \geqslant 0. \tag{3}$$

$$x \in E,\ Ax \in C_K \text{ and } f_0(x) \geqslant 0 \text{ imply } Ax = 0. \tag{4}$$

Then for any $y_0 \in F$ and any real number α, there exists an $x \in E$ satisfying

$$y_0 - Ax \in K, \qquad f_0(x) \leqslant \alpha, \tag{5}$$

if and only if

$$g \in K^0, \beta \geqslant 0 \text{ and } {}^tAg - \beta f_0 = 0 \text{ imply } g(y_0) - \beta\alpha \leqslant 1. \tag{6}$$

Proof. Denote by R, R^+, R^-, respectively, the real line, the set of all non-negative real numbers, and the set of all non-positive real numbers. Consider the topological vector space $F \times R$ with the product topology. Let $\mathscr{L}$ denote the vector subspace of $F \times R$ formed by all $(Ax, f_0(x))$ with $x \in E$. Since $f_0 \in {}^tA(F')$, there is a $g_0 \in F'$ such that $f_0 = {}^tAg_0$. Thus, $\mathscr{L} = \{(Ax, g_0(Ax)): x \in E\}$ is the kernel of the continuous linear form $(y, \eta) \rightarrow \eta - g_0(y)$ defined on $A(E) \times R$, so $\mathscr{L}$ is closed in $A(E) \times R$. As $A(E)$ is closed in F, it follows that $\mathscr{L}$ is a closed vector subspace of $F \times R$.

Let $\mathscr{K} = K \times R^+$. Since R^+ is a cone, it is easy to see that the asymptotic cone $C_{\mathscr{K}}$ of $\mathscr{K}$ is given by $C_{\mathscr{K}} = C_K \times R^+$. If $x \in E$ and $(Ax, g_0(Ax)) \in C_{\mathscr{K}} = C_K \times R^+$, then $f_0(x) = g_0(Ax) \geqslant 0$, so by hypothesis (4), we must have $Ax = 0$. Thus $C_{\mathscr{K}} \cap C_{\mathscr{L}} = C_{\mathscr{K}} \cap \mathscr{L}$ contains only the origin (0,0) of $F \times R$. By Dieudonné's result stated in Section 1, $\mathscr{K} + \mathscr{L}$ is closed in $F \times R$. Furthermore, we have by (3), $(Ax_0, f_0(x_0)) \in \mathscr{K} \cap \mathscr{L}$, so $\mathscr{K} + \mathscr{L}$ contains the origin (0,0) of $F \times R$. Therefore, by the bipolar theorem ([*6*], p. 248), $(\mathscr{K} + \mathscr{L})^{00} = \mathscr{K} + \mathscr{L}$.

Now the existence of an $x \in E$ satisfying (5) is equivalent to the relation $(y_0, \alpha) \in \mathscr{K} + \mathscr{L}$, which is the same as $(y_0, \alpha) \in (\mathscr{K} + \mathscr{L})^{00}$. Hence there exists an $x \in E$ satisfying (5), if and only if $(g, -\beta) \in (\mathscr{K} + \mathscr{L})^0$ implies $g(y_0) - \beta\alpha \leqslant 1$. This condition is equivalent to (6). Indeed, because $\mathscr{L}$ is a vector subspace of $F \times R$, we have $(g, -\beta) \in (\mathscr{K} + \mathscr{L})^0$ if and only if $(g, -\beta)(Ax, f_0(x)) = 0$ for all $x \in E$ and $(g, -\beta) \in \mathscr{K}^0 = K^0 \times R^-$. This completes the proof of Theorem 2.

THEOREM 3. *Let E, F be two separated locally convex topological vector spaces, and $A: E \rightarrow F$ a continuous linear map. Let P, Q be closed, locally compact, convex sets in E, F, respectively, such that at least one of P, Q is a cone, and $0 \in A(P) + Q$. Suppose that the following condition is fulfilled:*

$$x \in C_P \text{ and } Ax \in -C_Q \text{ imply } x = 0. \tag{7}$$

Then, for any $y_0 \in F$, there exists an $x \in E$ satisfying

$$x \in P, \qquad y_0 - Ax \in Q, \tag{8}$$

if and only if

$$g \in Q^0 \text{ and } {}^tAg \in P^0 \text{ imply } g(y_0) \leqslant 1. \tag{9}$$

Proof. Define the continuous linear map $\mathscr{A}: E \rightarrow E \times F$ by $\mathscr{A}(x) = (-x, Ax)$ for $x \in E$. $\mathscr{A}(E)$ is closed in $E \times F$, because it is the kernel of the continuous linear map $(x, y) \rightarrow Ax + y$ from $E \times F$ into F. Consider the closed, locally compact, convex set $\mathscr{P} = P \times Q$ in $E \times F$. As $0 \in A(P) + Q$, we have $\mathscr{A}(E) \cap \mathscr{P} \neq \varnothing$. It is easy to see that the asymptotic cone $C_{\mathscr{P}}$ of $\mathscr{P} = P \times Q$ is given by $C_{\mathscr{P}} = C_P \times C_Q$. By hypothesis (7), $(-x, Ax) \in C_{\mathscr{P}} = C_P \times C_Q$ implies $x = 0$. Hence $\mathscr{A}(E) \cap C_{\mathscr{P}}$ contains only the origin (0,0) of $E \times F$. By Theorem 1, there exists an $x \in E$ satisfying

$$(0, y_0) - \mathscr{A}(x) \in \mathscr{P}, \tag{10}$$

if and only if

$$(f,g) \in \mathscr{P}^0 \text{ and } {}^t\mathscr{A}(f,g) = 0 \text{ imply } (f,g)(0,y_0) \leq 1. \tag{11}$$

Clearly (10) is the same as (8). Since one of P, Q is a cone, the polar $\mathscr{P}^0$ of $\mathscr{P}$ in $(E \times F)' = E' \times F'$ is given by $\mathscr{P}^0 = P^0 \times Q^0$. Condition (11) states that $f \in P^0$, $g \in Q^0$ and ${}^tAg - f = 0$ imply $g(y_0) \leq 1$, so it is the same as condition (9). Theorem 3 is thus proved.

Remark 2. Theorem 3 will cease to be valid if hypothesis (7) is dropped. This can be seen from the example discussed in Remark 1, if the set K considered there is taken as Q, and if R^3 is taken as P.

3. Dual Extremal Problems

Using Theorem 2, we can prove the following result.

THEOREM 4. *Let E, F be two separated, locally convex, topological vector spaces. Let $A: E \to F$ be a continuous linear map such that $A(E)$ is closed in F. Let K be a closed, locally compact, convex set in F, and $f_0 \in {}^tA(F')$ such that conditions* (3), (4) *are fulfilled. Let $y_0 \in A(E) + K$. Then:*

(a) *$f_0(x)$ is bounded from below on the set $\{x \in E: y_0 - Ax \in K\}$, if and only if there exist $g \in F'$ and a real number β satisfying*

$$g \in K^0, \qquad \beta > 0, \qquad {}^tAg = \beta f_0. \tag{12}$$

(b) *If there exist $g \in F'$ and a real number β satisfying* (12), *then the minimum of $f_0(x)$ over the set $\{x \in E: y_0 - Ax \in K\}$ is attained, and is equal to the supremum of $[g(y_0) - 1]/\beta$, when $(g,\beta) \in F' \times R$ varies under condition* (12):

$$\operatorname{Min}\{f_0(x): y_0 - Ax \in K\} = \operatorname{Sup}\left\{\frac{g(y_0) - 1}{\beta} : g \in K^0, \beta > 0, {}^tAg = \beta f_0\right\}. \tag{13}$$

Proof. To prove the theorem, it suffices to verify the following statements (i) and (ii).

(i) *If no $(g,\beta) \in F' \times R$ can satisfy* (12), *then $f_0(x)$ is not bounded from below on the set $\{x \in E: y_0 - Ax \in K\}$.*

Since $y_0 \in A(E) + K$, we can find $x_1 \in E$ such that $y_0 - Ax_1 \in K$. Then $g \in K^0$ and ${}^tAg = 0$ imply $g(y_0) = g(y_0 - Ax_1) \leq 1$. As no $(g,\beta) \in F' \times R$ satisfies (12), this implication means that condition (6) is verified for every real number α. By Theorem 2, for every $\alpha \in R$, there exists an $x \in E$ satisfying (5). Hence $f_0(x)$ is not bounded from below on $\{x \in E: y_0 - Ax \in K\}$. This proves (i).

(ii) *If there exists $(g,\beta) \in F' \times R$ satisfying* (12), *then the minimum of $f_0(x)$ over the set $\{x \in E: y_0 - Ax \in K\}$ is attained, and* (13) *holds.*

Let M denote the set of those real numbers α for which (5) has a solution $x \in E$. By Theorem 2, M coincides with the set of those $\alpha \in R$ for which condition (6) is satisfied. Since $y_0 \in A(E) + K$, we can choose $x_1 \in E$ such that $y_0 - Ax_1 \in K$. Then $g \in K^0$ and ${}^tAg = 0$ imply $g(y_0) = g(y_0 - Ax_1) \leqslant 1$. Consequently, under the assumption $y_0 \in A(E) + K$, condition (6) is equivalent to the following one:

$$g \in K^0,\ \beta > 0 \text{ and } {}^tAg = \beta f_0 \text{ imply } \alpha \geqslant \frac{g(y_0) - 1}{\beta}. \tag{14}$$

Hence M is the set of all $\alpha \in R$ having property (14). It follows that $\mathrm{Min}_{\alpha \in M}\, \alpha$ exists and is equal to the supremum of $[g(y_0) - 1]/\beta$ when $(g, \beta) \in F' \times R$ varies under condition (12). But according to the definition of M, $\mathrm{Min}_{\alpha \in M}\, \alpha$ is precisely the minimum of $f_0(x)$ over the set $\{x \in E: y_0 - Ax \in K\}$. This proves (ii).

In case the convex set K is a cone, Theorem 4 has the following simpler formulation:

THEOREM 5. *Let E, F be two separated, locally convex, topological vector spaces. Let $A: E \to F$ be a continuous linear map such that $A(E)$ is closed in F. Let K be a closed, locally compact, convex cone in F, and $f_0 \in {}^tA(F')$ such that the following condition is fulfilled:*

$$x \in E,\ Ax \in K \text{ and } f_0(x) \geqslant 0 \text{ imply } Ax = 0. \tag{15}$$

Let $y_0 \in A(E) + K$. Then:

(a) *$f_0(x)$ is bounded from below on the set $\{x \in E: y_0 - Ax \in K\}$, if and only if $f_0 \in {}^tA(K^0)$.*

(b) *If $f_0 \in {}^tA(K^0)$, then the minimum of $f_0(x)$ over the set $\{x \in E: y_0 - Ax \in K\}$ is attained, and*

$$\mathrm{Min}\,\{f_0(x): y_0 - Ax \in K\} = \mathrm{Sup}\,\{g(y_0): g \in K^0,\ {}^tAg = f_0\}. \tag{16}$$

Proof. Since K is a cone, condition (3) is automatically satisfied, and condition (4) is the same as (15). K^0 is, like K, a cone, so the existence of $(g, \beta) \in F' \times R$ satisfying (12) is equivalent to the relation $f_0 \in {}^tA(K^0)$. Let

$$\sigma = \mathrm{Sup}\,\{h(y_0): h \in K^0,\ {}^tAh = f_0\},$$

$$\tau = \mathrm{Sup}\left\{\frac{g(y_0) - 1}{\beta}: g \in K^0,\ \beta > 0,\ {}^tAg = \beta f_0\right\}.$$

To derive Theorem 5 from Theorem 4, it suffices to verify that if $f_0 \in {}^tA(K^0)$, then $\sigma = \tau$.

Assume $f_0 \in {}^tA(K^0)$. By Theorem 4, the supremum τ is finite. For any $\epsilon > 0$, we can find $(g, \beta) \in F' \times R$ satisfying (12) such that $[g(y_0) - 1]/\beta > \tau - \epsilon$.

If we take $h = g/\beta$, then $h \in K^0$, ${}^tAh = f_0$ and $\sigma \geqslant h(y_0) = g(y_0)/\beta > \tau - \epsilon$. This shows that $\sigma \geqslant \tau$.

On the other hand, if $h \in K^0$ and ${}^tAh = f_0$, then for any $\beta > 0$, $g = \beta h$ and β will satisfy (12), so $h(y_0) = [g(y_0)]/\beta \leqslant \tau + (1/\beta)$. As β can be arbitrarily large, we must have $h(y_0) \leqslant \tau$ for every $h \in K^0$ satisfying ${}^tAh = f_0$. This proves $\sigma \leqslant \tau$ and therefore, $\sigma = \tau$.

THEOREM 6. *Let E, F be two separated, locally convex, topological vector spaces, and $A : E \to F$ a continuous linear map. Let P, Q be closed, locally compact, convex sets in E, F, respectively, such that at least one of them is a cone. Suppose that $f_0 \in E'$ satisfies the following two conditions:*

$$\text{There exists an } x_0 \in -P \text{ such that } Ax_0 \in Q \text{ and } f_0(x_0) \geqslant 0; \tag{17}$$

$$x \in -C_P,\ Ax \in C_Q \text{ and } f_0(x) \geqslant 0 \text{ imply } x = 0. \tag{18}$$

Let $y_0 \in A(P) + Q$. Then:

(a) *$f_0(x)$ is bounded from below on the set $\{x \in E : x \in P, y_0 - Ax \in Q\}$, if and only if there exist $g \in F'$ and $\beta \in R$ satisfying*

$$g \in Q^0, \qquad \beta > 0, \qquad {}^tAg - \beta f_0 \in P^0. \tag{19}$$

(b) *If there exists $(g, \beta) \in F' \times R$ satisfying* (19), *then the minimum of $f_0(x)$ over the set $\{x \in E : x \in P, y_0 - Ax \in Q\}$ is attained, and is equal to the supremum of $[g(y_0) - 1]/\beta$, when $(g, \beta) \in F' \times R$ varies under condition* (19):

$$\operatorname{Min}\{f_0(x) : x \in P, y_0 - Ax \in Q\} \tag{20}$$

$$= \operatorname{Sup}\left\{\frac{g(y_0) - 1}{\beta} : g \in Q^0, \beta > 0, {}^tAg - \beta f_0 \in P^0\right\}.$$

Proof. Consider the topological product vector space $E \times F$, and the continuous linear map $\mathscr{A} : E \to E \times F$ defined by $\mathscr{A}(x) = (-x, Ax)$ for $x \in E$. $\mathscr{A}(E)$ is closed in $E \times F$, since it is the kernel of the continuous linear map $(x, y) \to Ax + y$ from $E \times F$ into F. The transpose ${}^t\mathscr{A} : E' \times F' \to E'$ of $\mathscr{A}$ is given by ${}^t\mathscr{A}(h, g) = {}^tAg - h$ for $(h, g) \in E' \times F'$. Clearly, ${}^t\mathscr{A}(E' \times F') = E'$, and, therefore, $f_0 \in {}^t\mathscr{A}(E' \times F')$. Let $\mathscr{P} = P \times Q$, which is a closed, locally compact, convex set in $E \times F$. Condition (17) means that there is an $x_0 \in E$ satisfying $\mathscr{A}(x_0) \in \mathscr{P}$ and $f_0(x_0) \geqslant 0$. Because at least one of P, Q is a cone, we have $C_{\mathscr{P}} = C_P \times C_Q$. Thus condition (18) means that $x \in E$, $\mathscr{A}(x) \in C_{\mathscr{P}}$ and $f_0(x) \geqslant 0$ imply $x = 0$. Since $y_0 \in A(P) + Q$, we have $(0, y_0) \in \mathscr{A}(E) + \mathscr{P}$. Again, because one of P, Q is a cone, we have $\mathscr{P}^0 = P^0 \times Q^0$.

In order to derive the present theorem from Theorem 4, it suffices to make the following observation. First, the relation $(0, y_0) - \mathscr{A}(x) \in \mathscr{P}$ is equivalent to $x \in P$ and $y_0 - Ax \in Q$. Secondly, the existence of $(h,g) \in E' \times F'$ and $\beta \in R$ satisfying

$$(h,g) \in \mathscr{P}^0, \qquad \beta > 0 \text{ and } {}^t\mathscr{A}(h,g) = \beta f_0, \tag{21}$$

is equivalent to the existence of $(g,\beta) \in F' \times R$ satisfying (19). Finally, the supremum on the right side of (20) is precisely the supremum of $[(h,g)(0,y_0) - 1]/\beta$, when $(h,g) \in E' \times F'$ and $\beta \in R$ vary under condition (21).

If both P, Q are cones, then Theorem 6 has the following simpler formulation.

THEOREM 7. *Let E, F be two separated, locally convex, topological vector spaces, and $A: E \to F$ a continuous linear map. Let P, Q be closed, locally compact, convex cones in E, F, respectively. Let $f_0 \in E'$ be such that the following condition is fulfilled:*

$$x \in -P,\ Ax \in Q \text{ and } f_0(x) \geqslant 0 \text{ imply } x = 0. \tag{22}$$

Let $y_0 \in A(P) + Q$. Then:

(a) *$f_0(x)$ is bounded from below on the set, $\{x \in E: x \in P, y_0 - Ax \in Q\}$, if and only if there exists $g \in F'$ satisfying*

$$g \in Q^0, \qquad {}^tAg - f_0 \in P^0. \tag{23}$$

(b) *If there exists a $g \in F'$ satisfying* (23), *then the minimum of $f_0(x)$ over the set $\{x \in E: x \in P, y_0 - Ax \in Q\}$ is attained, and is equal to the supremum of $g(y_0)$, when g varies under condition* (23):

$$\operatorname{Min}\{f_0(x): x \in P, y_0 - Ax \in Q\}$$
$$= \operatorname{Sup}\{g(y_0): g \in Q^0, {}^tAg - f_0 \in P^0\}. \tag{24}$$

Proof. Since P, Q are closed convex cones, condition (17) is automatically verified, while condition (18) is the same as (22). Because P^0, Q^0 are also cones, the existence of $(g,\beta) \in F' \times R$ satisfying (19) is equivalent to the existence of $g \in F'$ satisfying (23). Let

$$\sigma = \operatorname{Sup}\{h(y_0): h \in Q^0, {}^tAh - f_0 \in P^0\},$$

$$\tau = \operatorname{Sup}\left\{\frac{g(y_0) - 1}{\beta}: g \in Q^0, \beta > 0, {}^tAg - \beta f_0 \in P^0\right\}.$$

To derive Theorem 7 from Theorem 6, it suffices to verify that, if there exists $g \in F'$ satisfying (23), then $\sigma = \tau$. This is verified by an argument similar to that in the proof of Theorem 5.

Remark 3. In the Euclidean 3-space R^3, let P be the closed convex cone formed by all $(\xi_1,\xi_2,\xi_3) \in R^3$ satisfying $\xi_1 \geqslant 0$, $\xi_2 \geqslant 0$ and $\xi_1\xi_2 \geqslant \xi_3^2$. Let

$Q = -P$. Consider the linear map $A: R^3 \to R^3$ defined by $A(\xi_1, \xi_2, \xi_3) = (0, \xi_3, \xi_1)$. We regard R^3 as its own dual space by identifying $g = (\eta_1, \eta_2, \eta_3) \in R^3$ with the linear form $g(x) = \eta_1 \xi_1 + \eta_2 \xi_2 + \eta_3 \xi_3$ for $x = (\xi_1, \xi_2, \xi_3)$. Then P^0 is the set of all $(\eta_1, \eta_2, \eta_3) \in R^3$ such that $\eta_1 \leqslant 0$, $\eta_2 \leqslant 0$ and $\eta_3^2 \leqslant 4\eta_1\eta_2$. We have $Q^0 = -P^0$ and ${}^tA(\eta_1, \eta_2, \eta_3) = (\eta_3, 0, \eta_2)$. Let $y_0 = (0, -1, 0)$ and $f_0 = (0, 0, 1)$, i.e., the linear form $f_0(\xi_1, \xi_2, \xi_3) = \xi_3$. Then $x = (\xi_1, \xi_2, \xi_3)$ satisfies $x \in P$ and $y_0 - Ax \in Q$, if and only if $\xi_1 = \xi_3 = 0$ and $\xi_2 \geqslant 0$. For every such x, we have $f_0(x) = 0$. $g = (\eta_1, \eta_2, \eta_3)$ satisfies (23), if and only if $\eta_2 = 1$, $\eta_3 \leqslant 0$ and $\eta_3^2 \leqslant 4\eta_1$. For every such g, we have $g(y_0) = -1$. Thus, equality (24) is not verified. Here, Theorem 7 fails to apply, because condition (22) is not fulfilled.

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